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CENTER FOR MULTIVARIATE ANALYSIS Z D BAI ET AL. DEC 86
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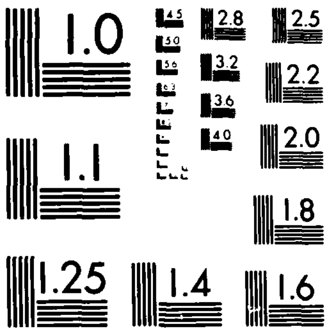
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 87-1018	2. GOVT ACCESSION NO	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On rate of convergence of equivariation linear prediction estimates of the number of signals and frequencies of multiple sinusoids		5. TYPE OF REPORT & PERIOD COVERED ~~Technical~~ Journal - December 1986
		6. PERFORMING ORG. REPORT NUMBER 86-38
7. AUTHOR(s) Z.D. Bai, P.R. Krishnaiah and L.C. Zhao		8. CONTRACT OR GRANT NUMBER(s) N00014-85-K-0292 (NAVY) F49620-85-C-0008 (Air Force)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Multivariate Analysis 515 Thackeray Hall University of Pittsburgh, Pittsburgh, PA 15260		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304 A5
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448		12. REPORT DATE December 1986
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Same as 11		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Estimation, Exponential Bounds, Frequencies, Number of Signals, Rate of Convergence, Signal Processing.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the authors investigated the rates of convergence of their estimates of frequencies and the number of signals under a signal processing model with multiple sinusoids.

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REFERENCES

[1] Z.D. Bai, P.R. Krishnaiah and L.C. Zhao, "On Rates of Convergence of Efficient Detection Criteria in Signal Processsign With White Noise", Technical Report No. 85-45, 1985. Center for Multivariate Analysis, University of Pittsburgh.

[2] Z.D. Bai, P.R. Krishnaiah and L.C. Zhao, " On Simultaneous Estimation of the Number of Signals and Frequencies Under a Model with Multiple Sinusoids", Techincal Report No. 86-37, 1986. Center for Multivariate Analysis, University of Pittsburgh.

AFOSR-TR- 87-1018

ON RATE OF CONVERGENCE OF EQUIVARIATION LINEAR PREDICTION ESTIMATES OF THE NUMBER OF SIGNALS AND FREQUENCIES OF MULTIPLE SINUSOIDS*

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Technical Report No. 86-38

December 1986

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*This work is supported by Contract N00014-85-K-0292 of the Office of Naval Research and Contract F49620-85-C-0008 of the Air Force Office of Scientific Research. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

1. INTRODUCTION

In a companion paper, the authors (see Bai, Krishnaiah and Zhao (1986)) considered the problem of estimation of the number of signals and the frequencies of these signals under a signal processing model with multiple sinusoids. The number of signals was estimated by using an information theoretic criterion. They have also established the strong consistency of their estimates. In this paper, we establish the rates of convergence of the above estimates of the number of signals and frequencies.

2. PRELIMINARIES AND STATEMENT OF PROBLEM

Consider the model

$$y(n) = \sum_{j=1}^{t_0} a_j \exp(i\omega_j n) + w(n), \quad n = 1,2,\ldots,N \tag{2.1}$$

where $i = \sqrt{-1}$, $\{a_j\}$ is a set of complex amplitudes, $\{\omega_j\}$ is a set of frequencies and $\{w(n)\}$ is the noise sequence of independent and identically distributed (i.i.d.) complex random variables with mean zero and $E|w(n)|^2 = \sigma^2 < \infty$. We assume that the frequencies $\omega_j \in (0,2\pi)$ are different from each other. Also, $y(n)$ is complex valued spatial sample observed at n-th array element. We will now describe the method of estimation of t_0, the number of signals, and ω_j's, the frequencies, considered in our earlier paper (Bai, Krishnaiah and Zhao(1986)). To determine t_0, it is assumed a prior that $t_0 \le T < \infty$. Let

$$S_t = \min\{\frac{1}{(N-t)} \sum_{n=t+1}^{N} |\sum_{\ell=0}^{t} b_\ell^{(t)} \overline{y(n-\ell)}|^2\} \tag{2.2}$$

for $t = 0,1,\ldots,T$, where the coefficients $b_\ell^{(t)}$ are subject to the restriction $\sum_{\ell=0}^{t} |b_\ell^{(t)}|^2 = 1$. Also, let

$$R_t = S_t + tC_N \tag{2.3}$$

where C_N satisfies the following restrictions:

$$\text{(i)} \quad \lim_{N\to\infty} C_N = 0$$

$$\text{(ii)} \quad \lim_{N\to\infty} \{\sqrt{N}\, C_N/\sqrt{\log\log N}\} = \infty. \tag{2.4}$$

The proposed estimate of t_0 is given by $\hat{t}_0$ where $\hat{t}_0$ is given by

$$R_{\hat{t}_0} = \min\{R_0, R_1, \ldots, R_T\} \tag{2.5}$$

Now, let $\hat{b} = (\hat{b}_0, \ldots, \hat{b}_{\hat{t}_0})'$ be such that

$$S_{\hat{t}_0} = \frac{1}{(N-\hat{t}_0)} \sum_{n=\hat{t}_0+1}^{N} \left| \sum_{\ell=0}^{\hat{t}_0} \hat{b}_\ell \overline{y(n-\ell)} \right|^2 \tag{2.6}$$

and let $\hat{\rho}_j \exp(i\hat{\omega}_j)$ be the roots of

$$\hat{H}(z) = \sum_{j=0}^{\hat{t}_0} \hat{b}_j z^j \tag{2.7}$$

where $\hat{\rho}_j \geq 0$ and $\hat{\omega}_j \in [0,2\pi)$ for $j = 1,\ldots,\hat{t}_0$. Then $\omega_1,\ldots,\omega_{t_0}$ were estimated with $\hat{\omega}_1,\ldots,\hat{\omega}_{\hat{t}_0}$ respectively. The strong consistency of $\hat{t}_0$ and $\hat{\omega}_j$'s was also established. In this paper, we are interested in establishing the rates of convergence of the above estimates.

Throughout this paper, A^* and A^- respectively denote transpose of the conjugate of A and general inverse of A.

3. CONVERGENCE RATE OF THE ESTIMATE OF THE NUMBER OF SIGNALS

In this section, we establish the rate of convergence of $\hat{t}_0$. Let

$$\hat{\gamma}_{\ell m}^{(t)} = \frac{1}{N-t} \sum_{n=t+1}^{N} y(n-\ell)\, \overline{y(n-m)}, \quad \ell,\ m=0,1,\ldots,t, \tag{3.1}$$

and $\hat{\Gamma}^{(t)} = (\hat{\gamma}_{\ell m}^{(t)})$. Also, let $\hat{\underset{\sim}{b}}_t = (\hat{b}_0^{(t)}, \hat{b}_1^{(t)}, \ldots, \hat{b}_t^{(t)})'$ be such that

$$S_t = \frac{1}{N-t} \sum_{n=t+1}^{N} \Big| \sum_{\ell=0}^{t} \hat{b}_\ell^{(t)} \overline{y(n-\ell)} \Big|^2. \tag{3.2}$$

Here we note that S_t is the smallest eigenvalue of the matrix $\hat{\Gamma}^{(t)}$.

Denote by $\delta_{\ell m}$ the Kronecker delta. Write

$$\gamma_{\ell m} = \sum_{j=1}^{t_0} |a_j|^2 e^{i(m-\ell)\omega_j} + \sigma^2 \delta_{\ell m}, \tag{3.3}$$

and $\Gamma^{(t)} = (\gamma_{\ell m})$ for $0 \leq \ell,\ m \leq t$. We can write $\hat{\gamma}_{\ell m}^{(t)}$ as

$$\hat{\gamma}_{\ell m}^{(t)} = \gamma_{\ell m} + J_1 + J_2 + J_3 + J_4 \triangleq \gamma_{\ell m} + \Delta_{N\ell m}^{(t)}, \tag{3.4}$$

$$\begin{aligned}
J_1 &= \sum_{\substack{j,k=1 \\ j\neq k}}^{t_0} a_j \bar{a}_k\, e^{i(m\omega_k - \ell\omega_j)} \frac{1}{N-t} \sum_{n=t+1}^{N} e^{in(\omega_j-\omega_k)} \\
J_2 &= \sum_{j=1}^{t_0} a_j\, e^{i(m-\ell)\omega_j} \frac{1}{N-t} \sum_{n=t+1}^{N} e^{i(n-m)\omega_j}\, \overline{w(n-m)} \\
J_3 &= \sum_{j=1}^{t_0} \bar{a}_j\, e^{i(m-\ell)\omega_j} \frac{1}{N-t}\, e^{-i(n-\ell)\omega_j} w(n-\ell) \\
J_4 &= \frac{1}{N-t} \sum_{n=t+1}^{N} [w(n-\ell)\overline{w(n-m)} - \delta_{\ell m}\sigma^2].
\end{aligned} \tag{3.5}$$

Write the eigenvalues of $\Gamma^{(t)}$ as $\lambda_0^{(t)} \geq \ldots \geq \lambda_t^{(t)} > 0$. From the structure of $\Gamma^{(t)}$, we have, for any $t \geq 1$,

$$\lambda_0^{(t)} \geq \lambda_0^{(t-1)} \geq \lambda_1^{(t)} \geq \lambda_1^{(t-1)} \geq \cdots \geq \lambda_{t-1}^{(t-1)} \geq \lambda_{(t)}^{(t)} \geq \sigma^2$$

and for $t \geq t_0$

$$\lambda_{t_0}^{(t)} = \lambda_{t_0+1}^{(t)} = \cdots = \lambda_t^{(t)} = \sigma^2$$

$$\lambda_{t_0-1}^{(t)} \geq \lambda_{t_0-1}^{(t_0-1)} > \sigma^2 \quad . \tag{3.6}$$

Now, let $\Delta = \lambda_{t_0-1}^{(t_0-1)} - \sigma^2$. Also, denote the eigenvalues of $\hat{\Gamma}(t)$ by $\hat{\lambda}_0^{(t)} \geq \hat{\lambda}_1^{(t)} \geq \cdots \geq \hat{\lambda}_t^{(t)}$. Then by Lemma 2.1 in Bai, Krishnaiah and Zhao (1985), we have

$$\sum_{\ell=0}^{t} (\hat{\lambda}_\ell^{(t)} - \lambda_\ell^{(t)})^2 \leq \mathrm{tr}(\hat{\Gamma}^{(t)} - \Gamma^{(t)})^2 \ . \tag{3.7}$$

Also, since $\hat{\lambda}_t^{(t)} = S_t$, we have

$$|S_t - \lambda_t^{(t)}| \leq \sqrt{\mathrm{tr}(\hat{\Gamma}^{(t)} - \Gamma^{(t)})^2} \quad t = 0,1,2,\ldots,T. \tag{3.8}$$

If $t > t_0$, then

$$\mathrm{tr}(\hat{\Gamma}^{(t)} - \Gamma^{(t)})^2 < \tfrac{1}{4} C_N^{\ 2} \tag{3.9}$$

and

$$\mathrm{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2 < \tfrac{1}{4} C_N^{\ 2} \tag{3.10}$$

implies that

$$\begin{aligned} R_t - R_{t_0} &= S_t - S_{t_0} + (t-t_0)C_N \\ &\geq C_N - [\sqrt{\mathrm{tr}(\Gamma^{(t)} - \hat{\Gamma}^{(t)})^2} + \sqrt{\mathrm{tr}(\hat{\Gamma}^{(t_0)} - \Gamma^{(t_0)})^2}] \\ &> 0 \, . \end{aligned} \tag{3.11}$$

Here we use the fact that $\lambda_t^{(t)} = \lambda_{t_0}^{(t_0)} = \sigma^2$. Hence $\hat{t}_0 \neq t$. If $t < t_0$, then

$$\mathrm{tr}(\Gamma^{(t)}-\hat{\Gamma}^{(t)})^2 < \tfrac{1}{4} C_N^{\,2} \tag{3.12}$$

and (3.10) implies that

$$\begin{aligned} R_t - R_{t_0} &= S_t - S_{t_0} - (t_0-t)C_N \\ &\geq \lambda_t^{(t)} - \sigma^2 - \{\sqrt{\mathrm{tr}(\hat{\Gamma}^{(t)}-\Gamma^{(t)})^2} + \sqrt{\mathrm{tr}(\hat{\Gamma}^{(t_0)}-\Gamma^{(t_0)})^2} + (t_0-t)C_N\} \\ &\geq \Delta - (t_0+2)C_N > 0 \end{aligned}$$

provided that $C_N < \Delta/(t_0+2)$. Hence $\hat{t}_0 \neq t$. Since $C_N \to 0$ for large N, we have $C_N < \Delta/(t_0+2)$. Therefore

$$P(\hat{t}_0 \neq t_0) \leq P(\bigcup_{t=0}^{T} [\mathrm{tr}(\hat{\Gamma}^{(t)}-\Gamma^{(t)})^2 \geq \tfrac{1}{4} C_N^{\,2}]). \tag{3.13}$$

<u>Theorem 3.1</u>. If we choose C_N satisfying the conditions

$$\lim_{N\to\infty} C_N = 0 \quad \text{and} \quad \lim_{N\to\infty} \sqrt{N}\, C_N = \infty$$

then $E|W(n)|^{2\eta} < \infty$, $\eta > 1$ implies

$$P(\hat{t}_0 \neq t_0) = O(N(N\, C_N)^{-\eta}), \text{ as } N \to \infty.$$

Also, $E\exp\{h|W(n)|^2\} < \infty$ for some $h > 0$, implies

$$P(\hat{t}_0 \neq t_0) = O(e^{-\delta N C_N^2}), \text{ for some } \delta > 0.$$

<u>Proof</u>. Both conclusions of this theorem follow from (3.13) and the expressions given in (3.5) and the well-known results of limit theorems concerning sums of independent random variables.

4. CONVERGENCE RATE OF FREQUENCY ESTIMATES

In this section, we establish the rate of convergence of the frequency estimates $\hat{\omega}_1,\ldots,\hat{\omega}_{\hat{t}_0}$.

Let $\underset{\sim}{b}_{t_0}$ be the eigenvector of $\Gamma^{(t_0)}$ corresponding to the smallest eigenvalue σ^2. Also, let

$$\hat{\underset{\sim}{b}}_{t_0} = \beta[I-(\hat{\Gamma}^{(t_0)}-S_{t_0}I)^*(\hat{\Gamma}^{(t)}-S_{t_0}I)(\hat{\Gamma}^{(t_0)}-S_{t_0}I)^*)^-(\hat{\Gamma}^{(t_0)}-S_{t_0}I)]\underset{\sim}{b}_{t_0}$$

$$= \beta[I-(\hat{\Gamma}^{(t_0)}-S_{t_0}I)^+(\hat{\Gamma}^{(t_0)}-S_{t_0}I)]\underset{\sim}{b}_{t_0} \tag{4.1}$$

where $\beta = \beta_N$ is a positive constant which normalizes $\hat{\underset{\sim}{b}}_{t_0}$ to unit length. Note that $\hat{\Gamma}^{(t_0)} \to \Gamma^{(t_0)}$ and $S_{t_0} \to \sigma^2$, with probability one. When N is large enough, we know that

$$[I-(\hat{\Gamma}^{(t_0)}-S_{t_0}I)^+(\hat{\Gamma}^{(t_0)}-S_{t_0}I)]\underset{\sim}{b}_{t_0} \neq 0 .$$

Hence $\hat{\underset{\sim}{b}}_{t_0}$ is well-defined. It is easy to verify that $\hat{\underset{\sim}{b}}_{t_0}$ is the eigenvector of $\hat{\Gamma}^{(t_0)}$ corresponding to the smallest eigenvalue S_{t_0}.

By the triangle inequality and (4.1), we have

$$|\hat{\underset{\sim}{b}}_{t_0}-\underset{\sim}{b}_{t_0}| \leq 2|(\hat{\Gamma}^{(t_0)}-S_{t_0}I)^+(\hat{\Gamma}^{(t_0)}-S_{t_0}I)\underset{\sim}{b}_{t_0}| . \tag{4.2}$$

Since $(\Gamma^{(t_0)}-\sigma^2 I)\underset{\sim}{b}_{t_0} = \underset{\sim}{0}$ and $|\underset{\sim}{b}_{t_0}| = 1$, we have

$$|(\hat{\Gamma}^{(t_0)}-S_{t_0}I)\underset{\sim}{b}_{t_0}|$$

$$= |(\hat{\Gamma}^{(t_0)}-\Gamma^{(t_0)} - (S_{t_0}-\sigma^2)I)\underset{\sim}{b}_{t_0}|$$

$$\leq [\mathrm{tr}(\hat{\Gamma}^{(t_0)}-\Gamma^{(t_0)})^2]^{\frac{1}{2}} + |S_{t_0}-\sigma^2|$$

Hence

$$|\hat{b}_{t_0}-b_{t_0}| \le 2|\hat{\lambda}_{t_0-1}^{(t_0)}-S_{t_0}|^{-1}\{[tr(\hat{\Gamma}^{(t_0)}-\Gamma^{(t_0)})^2]^{\frac{1}{2}} + |S_{t_0}-\sigma^2|\}. \qquad (4.3)$$

Now, let $\Delta = \lambda_{t_0-1}^{(t_0)} - \sigma^2 > 0$. Also, let $\eta > 0$ be a small number such that $\eta < \Delta/4(t_0+1)$. Then, for $t = t_0$, using (3.7) $|\hat{\gamma}_{j\ell}-\gamma_{j\ell}| \le \eta$, $j, \ell = 0,1,\ldots,t_0$, implies

$$[tr(\hat{\Gamma}^{(t_0)}-\Gamma^{(t_0)})^2]^{\frac{1}{2}} \le (t_0+1)\eta,$$

$$|S_{t_0}-\sigma^2| \le (t_0+1)\eta,$$

and

$$\hat{\lambda}_{t_0-1}^{(t_0)} - S_{t_0} \ge \lambda_{t_0-1}^{(t_0)} - \sigma^2 - 2(tr(\hat{\Gamma}^{(t_0)}-\Gamma^{(t_0)})^2)^{\frac{1}{2}}$$

$$\ge \Delta - 2(t_0+1)\eta > \frac{\Delta}{2}. \qquad (4.4)$$

Thus, by (4.2) - (4.4),

$$|\hat{b}_{t_0} - b_{t_0}| < 8(t_0+1)\Delta^{-1}\eta. \qquad (4.5)$$

For any $\varepsilon > 0$, let $\eta = \varepsilon\Delta/[8(t_0+1)]$. Then $|\hat{\gamma}_{\ell m}-\gamma_{\ell m}| \le \eta$, $\ell, m = 0,1,\ldots,t_0$ implies that

$$|\hat{b}_{t_0} - b_{t_0}| < \varepsilon. \qquad (4.6)$$

Therefore

$$P(|\hat{b}_{t_0}-b_{t_0}| \ge \varepsilon) \le \sum_{\ell=0}^{t_0}\sum_{m=0}^{t_0} P(|\hat{\gamma}_{\ell m}-\gamma_{\ell m}| \ge \eta). \qquad (4.7)$$

Now, define $|b_1 - b_2|$ as usual if b_1, b_2 have common dimensionality and ∞ otherwise. Also, $\hat{b} = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_{\hat{t}_0})'$. From (4.7) we have

$$P(|\hat{b}-b_{t_0}| \ge \varepsilon) \le P(|\hat{b}_{t_0}-b_{t_0}| \ge \varepsilon) + P(\hat{t}_0 \ne t_0) \qquad (4.8)$$

From (4.7), (4.8) and what has been proved in Section 3 we obtain the following theorem.

<u>Theorem 4.1.</u> Let C_N be chosen satisfying the conditions

$$\text{(i)} \quad \lim_{N\to\infty} C_N = 0 \qquad \text{(ii)} \quad \lim_{N\to\infty} \sqrt{N}\, C_N/\sqrt{\log\log N} = \infty.$$

Then $E|w(n)|^{2\eta} < \infty$, $\eta > 1$ implies

$$P(|\hat{\underset{\sim}{b}} - \underset{\sim}{b}_{t_0}| \geq \varepsilon) = O(N(NC_N)^{-\eta}). \tag{4.9}$$

Also,

$$E \exp\{h|w(n)|^2\} < \infty \text{ for some } h > 0$$

implies

$$P(|\hat{\underset{\sim}{b}} - \underset{\sim}{b}_{t_0}| > \varepsilon) = O(\exp\{-bNC_N^2\}) \text{ for some } b > 0 \tag{4.10}$$

Since the roots of the polynomial are continuous functions of coefficients of the polynomial, we have

<u>Theorem 4.2</u>. Suppose $\hat{\rho}_j e^{i\hat{\omega}_j}$, $j = 0,1,\ldots,\hat{t}_0$ are the roots of $\sum_{\ell=0}^{\hat{t}_0} \hat{b}_\ell z^\ell$ $\hat{\omega}_j \in [0,2\pi)$ and $\hat{\omega}_j$'s are arranged in increasing order. Also, $\exp(i\omega_j)$, $j=0,1,\ldots,t_0$ are ranked in the increasing order of ω_j. In addition, let $\hat{\underset{\sim}{z}} = (\hat{\rho}_0 e^{i\hat{\omega}_0}, \hat{\rho}_1 e^{i\hat{\omega}_1}, \ldots, \hat{\rho}_{\hat{t}} e^{i\hat{\omega}_{\hat{t}_0}})'$ and $\underset{\sim}{z} = (e^{i\omega_0}, \ldots, e^{i\omega_{t_0}})'$, we choose C_N satisfying the conditions

$$\text{(i)} \quad \lim_{N\to\infty} C_N = 0, \quad \text{(ii)} \quad \lim_{N\to\infty} \sqrt{N}\, C_N/\sqrt{\log\log N} = \infty.$$

Then $E|w(n)|^{2\eta} < \infty$, $\eta > 1$ implies

$$P(|\hat{\underset{\sim}{z}} - \underset{\sim}{z}| \geq \varepsilon) = O(N(C_N N)^{-\eta}). \tag{4.11}$$

Aslo,

$$E \exp\{h|w(n)|^2\} < \infty \text{ for some } h > 0 \tag{4.12}$$

implies

$$P(|\hat{\underset{\sim}{z}}-\underset{\sim}{z}|\geq\varepsilon) = O(\exp\{-bNC_N^2\}) \text{ for some } b > 0 \tag{4.13}$$

We note that (4.12) and (4.13) are respectively equivalent to the statements that $P(\max_{1\leq\ell\leq\hat{t}_0} |\hat{\rho}_\ell-1|\geq \varepsilon)$ is of order in (4.9) and $P(\max_{1\leq\ell\leq\hat{t}_0\wedge t_0} |\hat{\omega}_\ell-\omega_\ell| \geq \varepsilon)$ is of order in (4.10) where a $\wedge$ b denotes the minimum of a and b.

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